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Thermal stresses induced by a point heat source in a circular plate by quasi-static approach

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Abstract The present paper deals with the determination of quasi-static thermal stresses due to an instantaneous point heat source of strength g_{pi} situated at certain circle along the radial direction of the circular plate and releasing its heat spontaneously at time $t = \tau$. A circular plate is considered having arbitrary initial temperature and subjected to time dependent heat flux at the fixed circular boundary of $r = b$. The governing heat conduction equation is solved by using the integral transform method, and results are obtained in series form in terms of Bessel functions. The mathematical model has been constructed for copper material and the thermal stresses are discussed graphically. [doi:10.1063/2.1103107]

Keywords thermoelastic problem, thermal stresses, non-homogeneous heat conduction equation, heat generation

The steady-state thermal stresses in a circular plate subjected to an axi-symmetric temperature distribution on the upper surface with zero temperature on the lower surface and the circular edge has been considered by Nowacki.[1] Roy Choudhary[2] succeeded in determining the quasi-static thermal stresses in a thin circular plate subjected to transient temperature along the circumference of a circle over the upper face with lower face at zero temperature and fixed circular edge thermally insulated. Wankhede[3] determined the quasi-static thermal stresses in a thin circular disk subjected to arbitrary initial temperature on the upper surface with lower surface at zero temperature and fixed circular edge thermally insulated. Recently Deshmukh et al.[4] determined the thermal stresses in a hollow circular disk arising from internal heat generation. The present paper deals with the determination of displacement and thermal stresses in a circular plate defined as $0 \le r \le b$ arising from internal heat generation. A circular plate is considered having arbitrary initial temperature and subjected to time dependent heat flux at the fixed circular boundary of $r = b$. The governing heat conduction equation has been solved by using the integral transform method, and the results are obtained in series form in terms of Bessel functions. The results for displacement and stresses have been computed numerically and illustrated graphically.

To our knowledge no one has studied thermal stresses due to heat generation in a circular plate. This is thus thought a new and novel contribution. The results presented here will be useful for engineering problems, particularly in the determination of the state of stress in thin annular cylinders, which constitute essential elements for containers of hot gases or liquids, furnaces and similar facilities.

Consider a circular plate occupying space D defined by $0 \le r \le b$. Initially the plate is at an arbitrary temperature distribution of $F(r)$. A time dependent heat flux $Q(t)$ is applied on the fixed circular boundary ($r = b$). Heat generates within the solid at the rate of $\dfrac{g(r,t)}{k}$. Under these conditions, the displacement and thermal stresses in the plate due to internal heat generation are to be determined.

Following Deshmukh et al.[4] the differential equation governing the displacement potential function $\psi(r,\ t)$ is given as

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} = (1+\nu)\, a_t T. \tag{1}$$

The stress functions σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = \frac{-2\mu}{r}\frac{\partial\psi}{\partial r}, \tag{2}$$

$$\sigma_{\theta\theta} = -2\mu\frac{\partial^2\psi}{\partial r^2}, \tag{3}$$

while in each case the stress functions σ_{rz}, σ_{zz} and $\sigma_{\theta z}$ are zero within the plate in the plane state of the stress.

The temperature of the circular plate satisfies the following heat conduction equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{g(r,t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \tag{4}$$

with boundary conditions

$$k\frac{\partial T}{\partial r} = Q(t)\ \ (r = b,\quad t > 0), \tag{5}$$

and initial condition

$$T = F(r),\ \ (t = 0,\quad 0 \le r \le b), \tag{6}$$

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where k and α are thermal conductivity and thermal diffusivity of the material of the plate, respectively. Equations (1)-(6) constitute the mathematical formulation of the thermoelastic problem.

To obtain the expression for temperature $T(r,t)$, we introduce the finite Hankel transform over the variable r and its inverse transform defined by Öziski[5] as

$$\overline{T}(\lambda_n,t)=\int_{r'=0}^{b} r' K_0(\lambda_n,\ r')T(r',t)\mathrm{d}r', \tag{7}$$

$$T(r,t)=\sum_{n=1}^{\infty}\overline{T}(\lambda_n,\ t)K_0(\lambda_n,\ r)\ , \tag{8}$$

where

$$K_0(\lambda_m,r) = \frac{\sqrt{2}}{b}\frac{J_0(\lambda_m r)}{J_0(\lambda_m b)}, \tag{9}$$

and $\lambda_1,\ \lambda_2,\cdots$ are the positive roots of the transcendental equation

$$J_1(\lambda_m b) = 0, \tag{10}$$

$\mathrm{J}_n\,(X)$ is a Bessel function of the first kind of order n.

This transform satisfies the relation of

$$H\left[\frac{\partial^2 T}{\partial r^2}\ +\ \frac{1}{r}\frac{\partial T}{\partial r}\right] = -\lambda_n^2\overline{T}(\lambda_n,\ t). \tag{11}$$

On applying the finite Hankel transform defined in Eq. (7) and its inverse transform defined in Eq. (8) to Eq. (4), one obtains the expression for temperature as

$$\begin{aligned}T\,(r,t) = &\sum_{m=1}^{\infty} K_0(\lambda_m,r)\mathrm{e}^{-\alpha\lambda_m^2\cdot t'}\cdot\\ &\left\{\int_{r'=0}^{b} r' K_0(\lambda_m,r')F\,(r')\,\mathrm{d}r' + \frac{\alpha}{k}\int_{t'=0}^{t}\mathrm{e}^{\alpha\lambda_m^2\cdot t'}\cdot\right.\\ &\left[\int_{r'=0}^{b} r' K_0(\lambda_m,r')g(r',t')\mathrm{d}r' +\right.\\ &\left.\left. bK_0(\lambda_m,b)Q(t')\right]\mathrm{d}t'\right\}.\end{aligned} \tag{12}$$

Substituting Eq. (12) into Eq. (1), one obtains the displacement function ψ as

$$\begin{aligned}\psi = -\,(1+\nu)\,a_t &\sum_{m=1}^{\infty}\frac{1}{\lambda_m^2}K_0(\lambda_m,r)\mathrm{e}^{-\alpha\lambda_m^2\cdot t'}\cdot\\ &\left\{\int_{r'=0}^{b} r' K_0(\lambda_m,r')F\,(r')\,\mathrm{d}r' + \frac{\alpha}{k}\int_{t'=0}^{t}\mathrm{e}^{\alpha\lambda_m^2\cdot t'}\cdot\right.\\ &\left[\int_{r'=0}^{b} r' K_0(\lambda_m,r')g\,(r',t')\,\mathrm{d}r' +\right.\\ &\left.\left. bK_0(\lambda_m,b)Q\,(t')\right]\mathrm{d}t'\right\}.\end{aligned} \tag{13}$$

Substituting Eq. (13) into Eqs. (2) and (3), one obtains expressions for thermal stresses as

$$\begin{aligned}\sigma_{rr} = &-\frac{2\sqrt{2}}{b}\,(1+\nu)\,a_t\mu\cdot\\ &\sum_{m=1}^{\infty}\frac{1}{r\lambda_m}\frac{J_1(\lambda_m,r}{J_0(\lambda_m,r})\mathrm{e}^{-\alpha\lambda_m^2\cdot t'}\cdot\\ &\left\{\int_{r'=0}^{b} r' K_0(\lambda_m,r')F\,(r')\,\mathrm{d}r' + \frac{\alpha}{k}\int_{t'=0}^{t}\mathrm{e}^{\alpha\lambda_m^2\cdot t'}\cdot\right.\\ &\left[\int_{r'=0}^{b} r' K_0(\lambda_m,r')g\,(r',t')\,\mathrm{d}r' +\right.\\ &\left.\left. bK_0(\lambda_m,b)Q\,(t')\right]\mathrm{d}t'\right\},\end{aligned} \tag{14}$$

$$\begin{aligned}\sigma_{\theta\theta} = &-\frac{2\sqrt{2}}{b}\,(1+\nu)\,a_t\mu\cdot\\ &\sum_{m=1}^{\infty}\frac{1}{\lambda_m}\left(\lambda_m J_0(\lambda_m,r)\frac{J_1(\lambda_m,r)}{r}\right)\mathrm{e}^{-\alpha\lambda_m^2\cdot t'}\cdot\\ &\left\{\int_{r'=0}^{b} r' K_0(\lambda_m,r')F\,(r')\,\mathrm{d}r' + \frac{\alpha}{k}\int_{t'=0}^{t}\mathrm{e}^{\alpha\lambda_m^2\cdot t'}\cdot\right.\\ &\left.\left[\int_{r'=0}^{b} r' K_0(\lambda_m,r')g\,(r',t')\,\mathrm{d}r' + bK_0(\lambda_m,b)Q\,(t')\right]\mathrm{d}t'\right\}.\end{aligned} \tag{15}$$

Setting $F(r) = r^2$,
$g(r,t) = g_i\delta\,(r-r_1)\,\delta\,(t-\tau)\,,$
$Q\,(t) = \mathrm{e}^{-\omega t},\quad \alpha > 0,$
where r is the radius measured in meter, δ is the Dirac-delta function, $\omega > 0$. The heat source $g(r,t)$ is an instantaneous point heat source of strength $g_{pi} = 50$ J/m situated at certain circle along the radial direction of the plate and releasing its heat instantaneously at the time of $t\to\tau = 5$.

Radius of the circular plate $b = 1$ m,

Central circular paths of the circular plate $r_1 = 0.4$ m.

The numerical calculation has been carried out for a copper (pure) circular plate with the material properties defined as:

Thermal diffusivity $\alpha = 112.34\times 10^{-6}\ \mathrm{m^2s^{-1}}$.
Thermal conductivity $k = 386$ W/mk.
Density $\rho = 8\ 954\ \mathrm{kg/m^3}$.
Specific heat $c_p = 383$ J/kgK.
Poisson ratio $\nu = 0.35$.
Coefficient of linear thermal expansion
$a_t = 16.5\times 10^{-6}$/K,

Lamé constant $\mu = 26.67$.
$\lambda_1 = 3.831\ 7,\quad \lambda_2 = 7.015\ 6,\quad \lambda_3 = 10.173\ 5,$
$\lambda_4 = 13.323\ 7,\ \lambda_5 = 16.470,\quad \lambda_6 = 19.615\ 9,$
$\lambda_7 = 22.760\ 1,\ \lambda_8 = 25.903\ 7,\ \lambda_9 = 29.046\ 8,$
$\lambda_{10} = 32.18,$

are the roots of transcendental equation $J_1(\lambda\,b) = 0$. We set for convenience,

$$X = \frac{\sqrt{2}}{10^2 b},\quad Y = \frac{\sqrt{2}\,(1+\nu)\,a_t}{10^2 b},$$

$$Z = \frac{2\sqrt{2}\,(1+\nu)\,a_t\mu}{10^2 b},$$

here X, Y, Z are constants. The numerical calculation has been carried out with the help of computational

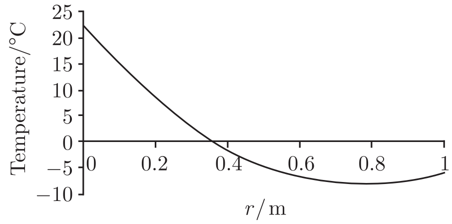

Fig. 1. Temperature distribution T/X .

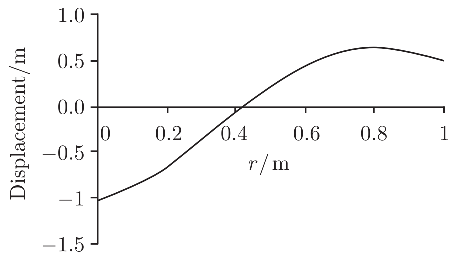

Fig. 2. Displacement function ψ/Y.

mathematical software MathCad 2000 and the graphs are plotted with the help of Excel (MS Office 2000).

From Fig. 1, it is observed that temperature reaches maximum at the center and then decreases towards the outer circular edge. After some time ($t = 5$), due to internal heat generation, the temperature of the copper plate increases proportionally.

From Fig. 2, it is observed that there is displacement occurring around the outer circular edge which is proportional to the temperature due to point heat source of strength g_p.

From Fig. 3, it is observed that the radial stress function σ_{rr} is zero at the outer circular boundary. As the heat source $g(r, t)$ is an instantaneous point heat source situated at certain circle within the central part ($r_1 = 0.5$) of the circular plate, we can observe the radial stress developing with compressive stresses around the circle of radius $r_1 = 0.5$ and the radial stress decreases towards the outer circular edge.

In Fig. 4, the angular stress function $\sigma_{\theta\theta}$ increases from the center to the outer circular edge. It reaches maximum at the center and develops a compressive stresses in the central part of $0 \leq r \leq 0.3$ with a tensile stresses in the annular region of $0.3 \leq r \leq 1$.

In this paper, we extend the work of Deshmukh et al.[4] to one dimensional non-homogeneous boundary value problem of heat conduction in a circular plate and determined the expressions of temperature, displacement and stresses due to internal heat generation.

As a special case, a mathematical model is constructed for copper (pure) circular plate with specified material properties. The heat source is an instantaneous point heat source of strength g_{pi} situated at certain circle along the radial and axial direction of the plate, and releases its heat instantaneously at the time of $t = \tau$. Owing to heat generation within the circular plate, the radial stress develops as compressive stresses,

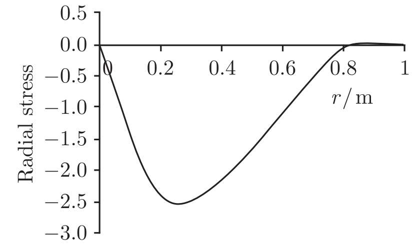

Fig. 3. Radial stress function σ_{rr}/Z.

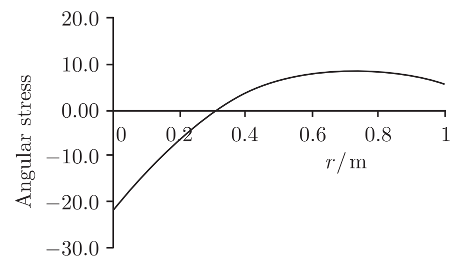

Fig. 4. Angular stress function $\sigma_{\theta\theta}/Z$.

whereas the angular stress develops with compressive stresses around the center and tensile stresses around the outer circular edge. Also it can be observed from the figures of temperature and displacement that the direction of heat flow and the direction of body displacement are opposite to each other. The results obtained here are useful for engineering problems, particularly in the determination of the state of stress in thin circular plates. Also any particular case of special interest can be derived by assigning suitable values to the parameters and functions in Eqs. (12)–(15).

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